AN INVERSE PROBLEM FROM SUB-RIEMANNIAN GEOMETRY

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To the memory of Robert B. Gardner

Abstract. The geodesics for a sub-Riemannian metric on a three-dimensional contact manifold M form a 1-parameter family of curves along each contact direction. However, a collection of such contact curves on M, locally equivalent to the solutions of a fourth-order ODE, are the geodesics of a sub-Riemannian metric only if a sequence of invariants vanish. The first of these, which was earlier identified by Fels, determines if the differential equation is variational. The next two determine if there is a well-defined metric on M and if the given paths are its geodesics.

Introduction

In this note, I will discuss the problem of recovering the geometric structure of a three-dimensional contact manifold with a sub-Riemannian metric from the geodesics for this metric. (Sub-Riemannian metrics are also known as Carnot-Carathéodory metrics.) Since all the results herein will be local in nature, the manifold may be taken to be an open set $U \in \mathbb{R}^3$ with contact form $dy - zdx$, and we may assume that on contact planes the metric has the form

$$g = Edx^2 + Fdxdz + Gdz^2,$$

where E, F, G are smooth functions on U such that g is positive definite. The geodesics, as constructed via the Griffiths formalism, form a collection of paths tangent to the contact structure, such that there is a 1-parameter family of distinct paths tangent to each contact direction at each point. Thus, part of the problem will be to determine which such collections of paths come from a sub-Riemannian metric.

As explained below, the paths are locally equivalent to the integral curves of a scalar fourth-order ODE. The *variational multiplier problem* for fourth-order ODE—i.e., the problem of characterizing equations which are, up to multiple, the Euler-Lagrange equations for a second-order Lagrangian—was solved by M. Fels [3]. Since sub-Riemannian geodesics arise

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as solutions of a variational problem, the present work is an extension of that of Fels; to avoid confusion, the notation of [3] will be used whenever possible.

1. Contact Path Geometries

In this section I will review the construction of sub-Riemannian geodesics, and define a G-structure canonically associated to the geodesics as paths.

Let M be an oriented three-manifold with contact distribution $\mathcal{D}$ and sub-Riemannian metric g. It is standard that one can associate to g a $SO(2)$-structure N inside the oriented coframe bundle $F(M)$, such that, for any coframing which is a local section of N, the forms $(\omega^1, \omega^2, \omega^3)$ of the coframing satisfy:

(i) ω^3 annihilates the contact planes;
(ii) $(\omega^1)^2 + (\omega^2)^2$ coincides with the metric on the contact planes; and,
(iii) $\omega^1 \wedge \omega^2$ gives the induced orientation.

Furthermore, one can choose N uniquely so that there is a connection form ϕ satisfying the structure equations[1]

$$\begin{aligned}
d\omega^1 &= \phi \wedge \omega^2 + (a_1\omega^1 + a_2\omega^2) \wedge \omega^3 \\
d\omega^2 &= -\phi \wedge \omega^1 + (a_2\omega^1 - a_1\omega^2) \wedge \omega^3 \\
d\omega^3 &= \omega^1 \wedge \omega^2 \\
d\phi &= K\omega^1 \wedge \omega^2 \mod \omega^3.
\end{aligned}$$

The functions a_1, a_2 are components of the torsion of g and K is called its scalar curvature.[2]

Every contact curve in M has a lift to N on which the forms ω^2 and ω^3 vanish. (In this way, we'll identify N with the space of contact directions on M.) Applying the Griffiths formalism [6] to find the integral curves in N of the Pfaffian system $\{\omega^2, \omega^3\}$ which are extremal curves for arclength $\int \omega^1$, we obtain the following characterization of sub-Riemannian geodesics[3]:

Proposition 1.1. *Let Z be the rank one affine subbundle of T^*N on which the canonical one-form is $\sigma = \omega^1 - x\omega^3$, $x \in \mathbb{R}$. (Forms on Z are pulled back via $\pi : Z \to N$.) Then*

[1]This result appears in [5], where it is attributed independently to Bryant-Hsu and to G. Wilkens.

[2] It's clear that taking the ω^i as an orthonormal coframe canonically associates to g with *Riemannian* metric $\widehat{g}$ on M, which induces g on $\mathcal{D}$, and defines a canonical foliation perpendicular to $\mathcal{D}$. The torsion tensor is the Lie derivative of $\widehat{g}$ along the leaves; if this vanishes, g descends to any (locally defined) quotient surface by foliation, and K is the Gauss curvature of the metric on that surface.

[3]See [8] for earlier derivations of the geodesics by other methods.

smooth geodesics are in 1-to-1 correspondence, via the submersion $Z \to M$, with integral curves of the Pfaffian system $\mathcal{F} = \{\theta_0, \theta_1, \theta_2, \theta_3\}$ on Z, where

$$\begin{aligned}
\theta_0 &= \omega^3 \\
\theta_1 &= \omega^2 \\
\theta_2 &= \phi - x\omega^1 \\
\theta_3 &= dx - a_1\omega_1 - a_2\omega_2.
\end{aligned}$$

Remark. In general, it is still an open question under what conditions all extremal curves for a given variational problem with differential constraints arise as projections of integral curves of the differential system formulated by Griffiths. For example, in sub-Riemannian geometry in dimension four, exceptional extremal curves exist which do not come from the Griffiths system. Essentially, this is because these *abnormal minimizers* [8] have few or no compactly supported variations that are tangent to the given distribution. However, by applying the regularity test given by Hsu [7], one can show that for a sub-Riemannian metric on a contact manifold, all geodesics arise via the Griffiths formalism. (Intuitively, compact variations exist because contact curves can be locally expressed in terms of an arbitrary function and its derivatives.)

Returning to the system $\mathcal{F}$ given above, let L be the line field on Z which is annihilated by $\mathcal{F}$. Integral curves of this line field push down via π to give a 1-parameter family of curves through each point of N, and push down to M to give a 1-parameter family of geodesics tangent to each contact direction. We will now generalize this situation, throwing away the metric:

Definition 1.2. *Let M^3 be a contact manifold. Let $\rho : P \to M$ be a fibration, with two-dimensional fibres, and L be a line field on P transverse to the two-dimensional fibres of ρ. Let $\mathcal{I}$ be the Pfaffian system on P which annihilates L and the fibres of ρ, and let $\mathcal{J}$ be the intersection of the retracting space* [2] *of $\mathcal{I}$ with the annihilator of L. Then (P, L, ρ) defines a contact path geometry on M if*

(i) for any vector $v \in L$, $\rho_*(v)$ is tangent to a contact plane;
(ii) the first derived system $\mathcal{I}'$ is one-dimensional at each point of P;
(iii) $\mathcal{J}' = \mathcal{I}$ at each point of P.

The last two conditions need explaining. Because of the transversality of L, $\mathcal{I}$ is two-dimensional. Because of condition (i), $\mathcal{I}'$ contains the pullback of any contact form on M. If $\mathcal{I}$ were integrable (i.e., $\mathcal{I}' = \mathcal{I}$, instead of being one-dimensional) then all paths through a given fibre $\rho^{-1}(x)$ would project down to a single contact curve on M, so that there would be only one path through $x \in M$. Condition (ii) implies that $\mathcal{J}$ is three-dimensional at each point. It is automatic that $\mathcal{I} \subset \mathcal{J}'$. If $\mathcal{J}$ were integrable, then integral surfaces of $\mathcal{J}$ would intersect $\rho^{-1}(x)$ in a 1-parameter family of curves; since each such surface would project down to a single contact curve in M, this would imply that there was only a 1-parameter family of paths through x.

Condition (ii) also implies that the three-dimensional distribution $\mathcal{D}$ containing L and the kernel of ρ_* is bracket-generating. That in turn guarantees, by Chow's theorem [1], that two arbitrary points in M can be connected by a piecewise smooth sequence of paths.

Proposition 1.3. *Given a contact path geometry we can construct, in a neighbourhood of any point $q \in P$, a coframe $(\sigma, \theta_0, \theta_1, \theta_2, \theta_3)$ such that*

1. *$v \in TP$ projects down to be a contact direction on M if and only if $\theta_0(v) = 0$*
2. *$\mathcal{I} = \{\theta_0, \theta_1\}$*
3. *$\mathcal{J} = \{\theta_0, \theta_1, \theta_2\}$*
4. *$L^\perp = \{\theta_0, \theta_1, \theta_2, \theta_3\}$*

Moreover, these forms satisfy

$$d\theta_i \equiv \theta_{i+1} \wedge \sigma \mod \theta_0, \dots, \theta_i, \qquad 0 \le i \le 2. \tag{1}$$

These will be called 0-adapted coframes for the contact path geometry.

Proof. Let $\rho(q) = x \in M$. On a neighbourhood V of x, there exists a contact form θ_0, and 1-forms σ, θ_1 such that $d\theta_0 \equiv \theta_1 \wedge \sigma \mod \theta_0$. Pull these forms back to $U = \rho^{-1}(V) \subset P$; we will shrink U when necessary. Since σ, θ_1 both restrict to be zero along the fibres of ρ, they cannot be independent modulo $\mathcal{I}$. Therefore we can arrange, by adding multiples of σ, that $\theta_1 \in \mathcal{I}$. (Note that now θ_1 is no longer the pullback of a form on V.). Since $\theta_0, \theta_1 \in L^\perp$, then $\sigma \notin L^\perp$. Since $\theta_0 \in \mathcal{I}'$, then $d\theta_1 \neq 0 \mod \mathcal{I}$.

Since $\theta_0, \theta_1, \sigma$ span an integrable system, then there will be a smooth 1-form θ_2 on U that $d\theta_1 \equiv \theta_2 \wedge \sigma \mod \mathcal{I}$. (Since $\mathcal{I}'$ is one-dimensional at each point, θ_2 is nonzero on U.) By adding multiples of σ, we can arrange that $\theta_2 \in L^\perp$, giving condition 3. Because $\mathcal{J}' \neq \mathcal{J}$,

there must be a nonzero 1-form θ_3 on U such that $d\theta_2 \equiv \theta_3 \wedge \sigma \mod \mathcal{J}$. We can similarly arrange that $\theta_3 \in L^\perp$. $\square$

Remark. The above proposition could also be proved just using the assumption that L is a line field on P and $\mathcal{I}, \mathcal{J}$ satisfy conditions (ii,iii) in Defn. 1.2. The contact structure and the submersion to M can be recovered from $\mathcal{I}'$ and the retracting space $\mathcal{C}(\mathcal{I}')$ respectively.

Corollary 1.4. *In some neighbourhood U of any given point $q \in P$, there exist coordinates x, y_0, y_1, y_2, y_3 such that, for some function F on U,*

$$\begin{aligned} \sigma &= -dx \\ \theta_0 &= dy_0 - y_1 dx \\ \theta_1 &= dy_1 - y_2 dx \\ \theta_2 &= dy_2 - y_3 dx \\ \theta_3 &= dy_3 - F(x, y_0, y_1, y_2, y_3) dx \end{aligned} \tag{2}$$

is a 0-adapted coframe. Consequently, paths in P are locally equivalent to the solutions of the fourth-order ODE

$$y'''' = F(x, y, y', y'', y'''). \tag{3}$$

Proof. The structure equations (1) enable us to apply the Goursat normal form theorem [2] to system $\mathcal{J}$. This gives $\mathcal{J} = \{\theta_0, \theta_1, \theta_2\}$, in terms of the forms defined here. Since $\mathcal{I} = \{\theta_0, \theta_1\}$, then $dx \notin L^\perp$, and so there exists some function F such that $dy_3 - F(x, y_0, y_1, y_2, y_3)dx = 0$ along the paths in U. $\square$

The set of 0-adapted coframes $(\sigma, \theta_0, \theta_1, \theta_2, \theta_3)$ for given contact path geometry forms a principal bundle over P, with structure group $G_0 \subset GL(5, \mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} a & * & * & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & * & a^{-1}b & 0 & 0 \\ 0 & * & * & a^{-2}b & 0 \\ 0 & * & * & * & a^{-3}b \end{pmatrix}.$$

This is precisely the G-structure that Fels associates to a fourth-order ODE up to contact transformation (cf. [3], Lemma 3.1). Since the path geometry can be recovered uniquely from the G-structure, we will treat the two notions as synonymous.

2. Variational and Sub-Riemannian Path Geometries

The goal of Cartan's method of equivalence [4] is, for a given G-structure, to find a subbundle, with reduced structure group, on which there exists a unique connection. Like the Levi-Civita connection in Riemannian geometry, this is typically obtained by fixing the value of all or part of the torsion of the connection. Then, invariants may be extracted from the remaining torsion or the curvature of the connection.

We begin with Fels' result for G_0-structures of coframes satisfying (1). This gives a reduction of structure to the subgroup $G_1 \subset G_0$ consisting of matrices of the form

$$\begin{pmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & a^{-1}b & 0 & 0 \\ 0 & 0 & 0 & a^{-2}b & 0 \\ 0 & 0 & 0 & 0 & a^{-3}b \end{pmatrix} \cdot \exp \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & c & 0 & 0 & 0 \\ 0 & 0 & \frac{4}{3}c & 0 & 0 \\ 0 & 0 & 0 & c & 0 \end{pmatrix}.$$

In terms of path geometry, the result is:

Theorem 1 (Fels [3])**.** *Let $B_0 \searrow P$ define a contact path geometry. Then there is a subbundle B_1 with three-dimensional structure group G_1, on which there exists a unique equivariant connection satisfying the following structure equations:*

(4a) $\quad d\sigma = \alpha \wedge \sigma + \theta_0 \wedge (T_1\theta_1 + T_2\theta_2 + T_3\theta_3) + \theta_1 \wedge (T_4\theta_2 + T_5\theta_3)$

(4b) $\quad d\theta_0 = \beta \wedge \theta_0 + \sigma \wedge \theta_1$

(4c) $\quad d\theta_1 = (\beta - \alpha) \wedge \theta_1 + \gamma \wedge \theta_0 + \sigma \wedge \theta_2$

(4d) $\quad d\theta_2 = (\beta - 2\alpha) \wedge \theta_2 + \frac{4}{3}\gamma \wedge \theta_1 + \sigma \wedge \theta_3$

(4e) $\quad d\theta_3 = (\beta - 3\alpha) \wedge \theta_3 + \gamma \wedge \theta_2 + \sigma \wedge (I_0\theta_0 + I_1\theta_1) + T_6\theta_0 \wedge \theta_1 + T_7\theta_0 \wedge \theta_2 + T_8\theta_1 \wedge \theta_2.$

[The one-forms α, β, γ are connection forms, and $I_0, I_1, T_1, \dots T_8$ are components of the torsion of the connection.]

Moreover, assuming P is locally defined by a fourth-order ODE (3)*, solutions of that ODE are critical curves for a second-order Lagrangian if and only if the relative invariants I_1 and T_5 both vanish identically on B_1. In that case, T_8 also vanishes.*

The essence of Fels' proof of the second statement is exhibiting a two-form on B_1,

$$\omega = m(\theta_0 \wedge \theta_3 - \theta_1 \wedge \theta_2),$$

where m is a non-zero function, such that ω is closed and G_1-invariant. (In fact, $d \log m = 3\alpha - 2\beta$, and, as Fels notes, the structure equations imply that that one-form is closed in the variational case.) It then follows that ω is the exterior derivative of the Poincaré-Cartan form associated to a Lagrangian on the space of 2-jets.

We will speak of a path geometry for which I_1, T_5, T_8 vanish identically as being *variational*.

Example 1. Consider the second-order Lagrangian $\int e^{-3y''} dx$, for which the Euler-Lagrangian equations are, up to multiple,

$$y'''' - 3(y''')^2 = 0.$$

The coframe (2) gives a section of the bundle B_0 defining the corresponding G_0-structure on $J^3(\mathbb{R}, \mathbb{R})$. This coframe may be modified to give the following section of the reduced structure B_1:

$$\begin{aligned}
\theta_0 &= dy_0 - y_1 dx \\
\theta_1 &= dy_1 - y_2 dx \\
\theta_2 &= dy_2 - y_3 dx - y_3\theta_1 + \tfrac{3}{10} y_3^2 \theta_0 \\
\theta_3 &= dy_3 - 3y_3 dy_2 - \tfrac{3}{10} y_3^2 \theta_1 + \tfrac{6}{5} y_3^3 \theta_0 \\
\sigma &= dx + \theta_1 - \tfrac{3}{5} y_3 \theta_0
\end{aligned} \tag{5}$$

Of course, the torsion satisfies $I_1 = T_5 = T_8 = 0$, but one also may compute[4] that $T_2 = \frac{12}{5} y_3$, $T_3 = \frac{3}{5}$ and $T_4 = -1$ along this section of B_1.

Example 2. (*sub-Riemannian geometry*) Let Z be the five-manifold of Proposition 1.1. It is easy to verify that the 1-forms given there, when rounded out by $\sigma = \omega^1 - x\omega^3$, form a 0-adapted coframe for the corresponding contact path geometry. We may adapt the coframe

[4] In order to evaluate the torsion components along a given section of B_1, one must determine the values of the connection forms in terms of the given coframe. To do this, begin with the $d\theta_0$ equation (4b), which determines β modulo θ_0. One may set $\beta = \beta_0 + b\theta_0$, where β_0 is any form satisfying (4b) and b is not yet determined. Then (4c) determines α and γ modulo θ_0, θ_1. In fact, one may set

$$\begin{aligned}
\alpha &= \alpha_0 + a\theta_0 + z\theta_1 \\
\gamma &= \gamma_0 - a\theta_1 + c\theta_0.
\end{aligned}$$

Now (4a) determines z while (4d),(4e) determine a, b and c.

to obtain a section of the reduced bundle $B_1 \searrow Z$:

$$
(6) \qquad \begin{aligned}
\theta_0 &= \omega^3 \\
\theta_1 &= \omega^2 \\
\theta_2 &= \phi - x\omega^1 + A\omega^3 \\
\theta_3 &= dx - a_1\omega^1 - (a_2 + A)\omega^2 + B\omega^3 \\
\sigma &= \omega^1 - \frac{3}{5}x\omega^3
\end{aligned}
\qquad \text{with} \qquad
\begin{aligned}
A &= \frac{1}{10}\left(a_2 + 3x^2 - 3K\right) \\
B &= \frac{1}{10}\left(s_2 - 3k_1 - 6a_1x - 21b_1\right),
\end{aligned}
$$

where K is the scalar curvature, and the b_i, s_i and k_i are defined on N by

$$
\left.\begin{aligned}
da_1 &\equiv 2a_2\phi + (s_1 + b_2)\omega^1 + (s_2 + b_1)\omega^2 \\
da_2 &\equiv -2a_1\phi + (s_2 - b_1)\omega^1 + (b_2 - s_1)\omega^2 \\
dK &\equiv k_1\omega^1 + k_2\omega^2
\end{aligned}\right\} \mod \omega^3
$$

Again, one may compute that $I_1 = T_5 = T_8 = 0$, confirming that the path geometry is variational, while $T_2 = 0$, $T_3 = \frac{3}{5}$, and $T_4 = -1$ for this coframe.

The fact that we obtained the same values for T_3 and T_4 as those from a general second-order Lagrangian hints at further relations among the torsion components. One uncovers one of these while deriving the *refined structure equations*:

Proposition 2.1. *Let B_1 be the canonical G_1-structure for a variational path geometry. Then there exist functions U_1, U_2 on B_1 such that the connection forms satisfy*

$$
(7) \qquad \begin{aligned}
d\alpha &= \tfrac{2}{3}d\beta \\
d\beta &= \sigma \wedge \gamma - \tau \wedge \theta_1 - 3\nu \wedge \theta_0 \\
d\gamma &\equiv \gamma \wedge \alpha - \tau \wedge \theta_2 - \nu \wedge \theta_1 \mod \theta_0
\end{aligned}
\qquad \textit{where} \qquad
\begin{aligned}
\tau &= T_1\theta_1 + T_2\theta_2 + T_3\theta_3 \\
\nu &= U_1\theta_1 + U_2\theta_2 - T_2\theta_3 + T_7\sigma
\end{aligned}.
$$

The torsion components satisfy $T_3 = -\frac{3}{5}T_4$ and

$$
\begin{aligned}
&(8) \qquad dT_1 \equiv T_1(2\alpha - 2\beta) - \tfrac{4}{3}T_2\gamma - 2U_1\sigma && \mod \theta_0, \theta_1, \theta_2, \theta_3 \\
&(9) \qquad dT_2 \equiv T_2(3\alpha - 2\beta) - \tfrac{2}{5}T_4\gamma - (T_1 + 2U_2)\sigma && \mod \theta_0, \theta_1, \theta_2, \theta_3 \\
&(10) \qquad dT_4 \equiv T_4(4\alpha - 2\beta) - \tfrac{5}{3}T_2\sigma && \mod \theta_0, \theta_1, \theta_2.
\end{aligned}
$$

The above equations indicate that T_4 is a *relative invariant* on B_1, i.e., it varies along the fibres only by scaling. Moreover, they indicate that the quadratic form $g = \sigma^2 - T_4\theta_1^2$ is

well-defined, up to multiple and modulo θ_0, on N. For, suppose v is a vector field on B_1 which is annihilated by $\sigma, \theta_0, \theta_1, \theta_2$. Then computing the Lie derivative of g gives

$$\begin{aligned}\mathcal{L}_v(g) &= 2\sigma \circ (v \lrcorner d\sigma) - 2T_4\theta_1 \circ (v \lrcorner d\theta_1) - (v \lrcorner dT_4)\theta_1^2 \\ &\equiv 2(v \lrcorner \alpha)\left[\sigma^2 - T_4\theta_1^2\right] \mod \theta_0.\end{aligned}$$

This quadratic form will be our candidate for a sub-Riemannian metric. Matters being so, we will say that a variational geometry is *nondegenerate* if $T_4 \neq 0$ everywhere,[5] and *definite* if T_4 is negative everywhere. Assuming the latter is the case, then we may normalize T_2 and T_4 to have the same values as in Example 2.

Proposition 2.2. *Let $B_1 \searrow P$ be a definite variational path geometry. Then there is a sub-bundle $B_2 \subset B_1$ on which*

$$T_2 = 0 \text{ and } T_4 = -1.$$

On B_2 there exist smooth functions W_0, W_1, W_2, G_0, G_1, G_2, G_3, and H such that

$$\beta = 2\alpha + W_0\theta_0 + W_1\theta_1 + W_2\theta_2 \tag{11}$$

$$\gamma = H\sigma - 3(G_0\theta_0 + G_1\theta_1 + G_2\theta_2 + G_3\theta_3). \tag{12}$$

Proof. Structure equations (9),(10) show that we may first pass to the sub-bundle where $T_4 = -1$ and then move along the fibres in a direction dual to γ to pass to the sub-bundle where $T_2 = 0$. Once there, these equations show that $\beta - 2\alpha$ and γ restrict to have the above form. Of course, (9) shows that $\frac{2}{5}H = T_1 + 2U_2$. ∎

On B_2, the structure equations (4) take the form

$$\begin{aligned} d\sigma &= \alpha \wedge \sigma + \theta_0 \wedge (T_1\theta_1 + \tfrac{3}{5}\theta_3) - \theta_1 \wedge \theta_2 \\ d\theta_0 &= 2\alpha \wedge \theta_0 + \sigma \wedge \theta_1 \\ d\theta_1 &= (\beta - \alpha) \wedge \theta_1 + \gamma \wedge \theta_0 + \sigma \wedge \theta_2 \\ d\theta_2 &= (\beta - 2\alpha) \wedge \theta_2 + \tfrac{4}{3}\gamma \wedge \theta_1 + \sigma \wedge \theta_3 \\ d\theta_3 &= (\beta - 3\alpha) \wedge \theta_3 + \gamma \wedge \theta_2 + I_0\sigma \wedge \theta_0 + T_6\theta_0 \wedge \theta_1 + T_7\theta_0 \wedge \theta_2. \end{aligned} \tag{13}$$

with β given by (11).

[5] Suppose a variational path structure has T_3 and T_4 identically zero; the refined structure equations show that $T_2 = 0$ also. Recall that the system which restricts to be zero along the fibres of $\rho : P \to M^3$ is spanned by $\sigma, \theta_0, \theta_1$. Since $d\sigma \equiv T_1\theta_0 \wedge \theta_1 \mod \sigma$, vectors that are in the kernel of σ push down to give a well-defined plane field on M. These planes intersect the contact planes in a *distinguished* family of contact directions, which are null lines with respect to g.

It's clear that the fibres of B_2 are one-dimensional, and the structure group of B_2 is simply $\mathbb{R}^*$. A element $\lambda \neq 0$ of this group acts on sections of B_2 by

$$g_\lambda \cdot (\sigma, \theta_0, \theta_1, \theta_2, \theta_3) = (\lambda\sigma, \lambda^2\theta_0, \lambda\theta_1, \theta_2, \lambda^{-1}\theta_3).$$

Structure equations (7) show that $g_\lambda^*\alpha = \alpha$, $g_\lambda^*\beta = \beta$, and $g_\lambda^*(\gamma) = \lambda^{-1}\gamma$. Then the action on the new torsion is clearly

$$\begin{aligned} g_\lambda \cdot (W_0, W_1, W_2) &= (\lambda^{-2}W_0, \lambda^{-1}W_1, W_2), \\ g_\lambda \cdot (H, G_0, G_1, G_2, G_3) &= (\lambda^{-2}H, \lambda^{-3}G_0, \lambda^{-2}G_1, \lambda^{-1}G_2, G_3). \end{aligned}$$

In particular, W_2, G_3, and the ratios $G_1 : W_0$ and $G_2 : W_1$ are invariant under the scaling action.

We should expect this scaling to be present, since two sub-Riemannian metrics which differ by a constant factor have the same geodesics and hence define the same path geometry. For purposes of constructing a specific metric, we will need to choose a section of B_2. Since $3\alpha - 2\beta$ is closed, integrals of this one-form comprise a canonical codimension-one foliation of B_2 which is transverse to fibres and invariant under the scaling action.

Definition 2.3. *A section of B_2 along which*

$$3\alpha - 2\beta = 0 \tag{14}$$

will be called a canonical section of B_2, or a canonical coframe on P. It follows from (11) *that*

$$\alpha = -2(W_0\theta_0 + W_1\theta_1 + W_2\theta_2) \tag{15}$$

along a canonical section.

One can check that the coframing constructed in Example 2 is a canonical coframe. Since such coframings are unique up to scale, it follows that if a path geometry comes from a sub-Riemannian metric, then in terms of a canonical coframe that metric must be $g = \sigma^2 + (\theta_1)^2$.

Proposition 2.4. *Let P be a definite variational path geometry on contact manifold M^3 and $(\sigma, \theta_0, \theta_1, \theta_2, \theta_3)$ a canonical coframe on P. Then $g = \sigma^2 + (\theta_1)^2$ gives a well-defined metric on the contact planes of M if and only if W_2 is identically zero on P.*

Proof. Let v be any vector field on P tangent to the fibres of the projection $\rho : P \to M$. Since v is annihilated by θ_0, θ_1 and σ,

$$\begin{aligned}\mathcal{L}_v(g) &\equiv (v \lrcorner \alpha)(\sigma)^2 + (v \lrcorner (\beta - \alpha))(\theta_1)^2 \mod \theta_0 \\ &\equiv -W_2(v \lrcorner \theta_2)\left(2(\sigma)^2 + (\theta_1)^2\right).\end{aligned}$$

□

Although the coframe (5) is not a section of B_2, it can be adjusted so that $T_2 = 0$, whereupon we see that W_2 is nonzero for Example 1.

For the rest of this section we will assume that W_2 is identically zero. It remains to be seen if the given paths on M^3—which are projections of the integral curves of the line field L—are the geodesics of the sub-Riemannian metric we have constructed. To investigate this further, we will need the torsion identities

$$G_3 = 0, \quad G_2 = W_1,$$

which result from computing $d(d\theta_1) = 0$ using the structure equations (13) and the equations (12),(14),(15) with $W_2 = 0$.

Remark. One might wonder if other identities hold among the remaining torsion coefficients G_0, G_1, H, I_0, T_1, T_6, T_7, W_0, W_1 as a result of our assumption that $W_2 = 0$. However, no further identities arise, and this is proved by showing that the exterior differential system defining a G-structure satisfying the structure equations on B_2 with $W_2 = 0$ is *involutive.*

Theorem 2. *Let P carry a variational and definite path geometry with invariant W_2 identically zero, and let $(\sigma, \theta_0, \theta_1, \theta_2, \theta_3)$ be a fixed canonical coframe on P. Then the paths in P project to be geodesics in M for the sub-Riemannian metric of Prop. 2.4 if and only if $G_1 = 2W_0$ identically on P.*

Proof. Let N be the quotient of P by the foliation by integral curves of the system $\mathcal{I}^{(1)} = \{\sigma, \theta_0, \theta_1, \theta_2\}$. Each contact curve in M has a unique lift to N as an integral curve of $\mathcal{I} = \{\theta_0, \theta_1\}$. Clearly, arclength is measured along these lifts by the integral of σ modulo $\mathcal{I}$. However, the form σ on P does not descend to be well-defined on N, as shown by

$$\begin{aligned}d\sigma &= \alpha \wedge \sigma + \theta_0 \wedge (T_1\theta_1 + T_3\theta_3) + T_4\theta_1 \wedge \theta_2 \\ &\equiv \tfrac{3}{5}\theta_0 \wedge \theta_3 \mod \Lambda^2\mathcal{I}^{(1)}.\end{aligned}$$

However, computing $d^2\theta_0 = 0$ shows that

$$dW_1 \equiv (G_1 - W_0)\sigma + \tfrac{1}{5}\theta_3 \mod \mathcal{I}, \tag{16}$$

and this, together with $d\theta_0 \equiv 0 \mod \Lambda^2\mathcal{I}^{(1)}$, shows that the 1-form

$$\widetilde{\sigma} = \sigma + 3W_1\theta_0$$

is well-defined on N.

Now arclength with respect to the metric may be measured on the integral curves of $\mathcal{I}$ by the Lagrangian $\int \widetilde{\sigma}$. We will apply the Griffiths formalism [6] to investigate which of these are geodesics for g. Then, we will try to find conditions under which these curves coincide with the projections of the paths in P under $\pi : P \to N$.

Let $\xi = \widetilde{\sigma} + x\theta_0 + y\theta_1$ on $Y = N \times \mathbb{R}^2$. Then one finds that the two-form $d\xi$ is of full rank on Y, except where $y = 0$. Accordingly, let $\xi = \widetilde{\sigma} + x\theta_0$ on $Z = N \times \mathbb{R}$. Now one computes that

$$d\xi \equiv (dx + (3G_1 - W_0)\sigma) \wedge \theta_0 + (\theta_2 + (x + 5W_1)\sigma) \wedge \theta_1 \mod \theta_0 \wedge \theta_1. \tag{17}$$

Let $\mathcal{K}$ be the rank four Pfaffian system on Z spanned by the four one-forms on the right in (17):

$$\mathcal{K} = \{\theta_0, \theta_1, \theta_2 + (x + 5W_1)\sigma, dx + (3G_1 - W_0)\sigma\}.$$

According to the Griffiths formalism, integral curves of $\mathcal{K}$ project to be extremal curves for $\int \widetilde{\sigma}$ on N. These coincide with the projections of the paths in P if and only if, in a neighbourhood U of each point of P, there is a local diffeomorphism $\varphi : U \to Z$ such that $\varphi^*\mathcal{K}$ coincides with $L^\perp = \{\theta_0, \theta_1, \theta_2, \theta_3\}$, the Pfaffian system on P which defines the paths. (The diffeomorphism would follow from the identification of paths with geodesics on N.) The form $\varphi^*(\theta_2 + (x + 5W_1)\sigma)$ belongs in $L^\perp$ if and only if $\varphi^*x = -5W_1$. Then, by (16),

$$\varphi^*(dx + (3G_1 - W_0)\sigma) \equiv (4W_0 - 2G_1)\sigma \mod L^\perp,$$

showing that $\varphi^*\mathcal{K} = L^\perp$ if and only if $G_1 = 2W_0$. □

3. Discussion

The results of the previous section may be surprising. For, one could reason that, once a path geometry is known to be variational, it must arise from a second-order Lagrangian, of the form

$$\int L(x, y, y', y'')dx,$$

satisfying the nondegeneracy condition $\partial^2 L/\partial(y'')^2 \neq 0$. Then L is of the form

$$L(x, y, y', y'') = \sqrt{E + Fy'' + G(y'')^2},$$

for some functions E, F, G of x, y, y', if and only if L satisfies a certain third-order ODE as a function of y''. In other words, it seems like only one extra condition must be satisfied in order for the Lagrangian to be length with respect to a sub-Riemannian metric. Instead, we find that two scalar conditions (in addition to the Fels variational condition) must hold in order for the metric to be well-defined and in order for its extremals to coincide with the given paths. (The reader should note that the above remark about involutivity implies that the condition $G_1 = 2W_0$ is independent from $W_2 = 0$.) It would be interesting to find examples of variational path geometries (equivalently, variational fourth-order ODE) for which $W_2 = 0$ but the extremals of the associated metric do not coincide with the given paths. Such examples must exist, again, because of the involutivity of $W_2 = 0$.

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